Notes For Mathematics & Physics Undergraduates

On Topology

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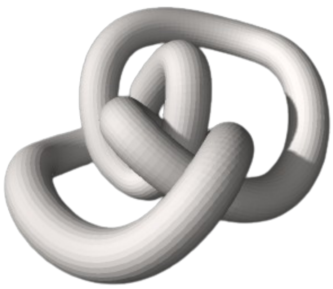


Contents

Introduction

In Einstein's general relativity the structure of space can change but not its topology. Topology is the property of something that doesn't change when you bend it or stretch it as long as you don't break anything.

Edward Witten

The syllabus for this course is usually decided by the department board (which are minimal for lecturing and maximal for examining). What is presented here contains some collection of some notes and results of Topology course, for undergraduates, mostly Mathematics and Theoretical Physics students. The last chapter contains exercises for practicing. Since it is presented this way, in my opinion, it would not be fair to set as a book-work although they could appear as problems. Also, I want from now to appreciate **very much** for contacting me about any corrections or possible improvements can be possibly done.Theses notes are written in LaTeX and should be available in PDF format from my home page:

https://rafid-hasan.com/

I

Metric Spaces

1.1

METRIC SPACES: (X, d)

For a set. The metric space is a set X ans a function d *called a distance function* or a metric on X such that:

1. $d(x, y) \geq 0 \, \forall x, y \in X$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x)$, e.g., $d(x, y) = ||x - y|| = ||(-1)(y - x)|| = ||y - x|| = d(y, x)$
4. $\forall d(x, y) \leq d(x, z) + d(z, y) \; x, y, z \in X$ **Triangle Inequality**
   E.g., $d(x, y) = ||x - y|| = ||(x - z) + (z - y)|| \leq ||x - z|| + ||z - y|| = d(x, z) + d(z, y)$

1.2

NORMS

$d : X \times X \longrightarrow [0, \infty)$

For a space. ||.|| is a **Norm** on Linear space - vector space if it satisfies the following conversions:

1. $||x|| \geq \; 0 \forall x \in X$
2. $||x|| = \vec{0} \iff x = \vec{0}$
3. $\forall_{x \in X} \forall_{\lambda \in \mathbb{C}} ||\lambda.x|| = |\lambda|.||x||$
4. $\forall_{x,y \in X} ||x + y|| \leq ||x|| + ||y||$

Note: A norm ||.|| defines a metric by: d(x,y) = ||x-y||

VECTOR SPACE

A vector space over $\mathbb{C}$ or $\mathbb{R}$ where $(X, +, ., \vec{0}, \mathbb{C})$

1. $\forall x + y \in X \; x, y \in X$
2. $\forall_{x \in X} \forall_{\lambda \in \mathbb{C}} \lambda x \in X$
3. $\mathbb{C} \ni 0.x = \vec{0} \in X$
4. $\lambda(x + y) = \lambda x + \lambda y$
5. x+y+z = (x+y) + z = x + (y+z)

1.3 INNER PRODUCT < , >

On a Complex/ Real a linear space X is a metric < , >: $X \times X \to \mathbb{C}$ *or* $\mathbb{R}$, such that:

1. $< x, x > \geq o, < x, x >= 0 \iff x = 0$
2. $<x, y+z> = <x,y> + <x,z>$
3. $< x, y >= \overline{< y, x >}$
4. $< x, \lambda y >= \lambda < x, y >$
   $< \lambda x, y >= \overline{< y, \lambda x >} = \overline{\lambda < y, x >} = \bar{\lambda}\overline{< y, x >} = \bar{\lambda} < x, y >$

NOTES:

- < , > define ||.|| by: $||x|| = \sqrt{< x, x >}$
- $||x + y||^2 = 2||x||^2 + 2||y||^2$
- $< x, y >^2 \leq < x, x > . < y, y >$ **Schwartz Inequality**

EXAMPLES ON INNER PRODUCT:

- $x, y \in \mathbb{R}^n$
  $< x, y >= \sum_{i=n}^{n} x_i \; y_i$
  $x = (x_1, ..., x_n)$
  $y = (y_1, ..., y_n)$
  $x_i \, , y_i \in \mathbb{R}$
- $x, y \in \mathbb{C}^n$
  $<x,y> = \sum \overline{x_i}\, y_i$
  (x/y) = $<A.x, y>$ *A positive defined metric for every bi-linear*
  Since A is positive defined $\iff < A.x, y >\geq 0$

NOTES:

- For the symmetry and permetation $A = A^T$,and $A = (A^T)^T$
- For A defined as 2×2 matrix:
  $P \; A \; P^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$x \in \mathbb{R}^n, ||.||_p$ is a norm defined of $+\infty > p \geq 1$:

$$||x||_P = \Big(\sum_{i=n}^{n} |x_i|^p\Big)^{1/p}$$

$||x||_2 = \sqrt{x_1^2 + ... + x_n^2}$

$||x||_1 = |x_1| + |x_2| + ... + |x_n|$

$|x| = c_1|x_1| + c_2|x_2| + ... + c_n|x_n|$

Note: If $||.||', ||, \dot{|}|$ are norms on $\mathbb{R}^n$ $\exists_{m,M>0}\forall_{x\in\mathbb{R}^n}$ in $||x|| \leq ||x||' \leq M.||x||$

1.4
BALLS

OPEN BALLS:

An open ball B(x,r) with center x and radius r in a metric space (X,d) is the set:

$B(x,r) = \{y \in X : d(x,y) < r\}$

CLOSE BALLS:

A close ball B(x,r) with center x and radius r in a metric space (X, d) is the set:

$B(x,r) = \{y \in X : d(x,r) \leq r\}$, and we denote it: $\overline{B}(x,y)$

Examples:

$\mathbb{R}^2$ and $B((0,0),1)$:

$||x||_2 < 1 \iff \sqrt{x_1^2 + x_2^2} < 1$

$||x||_1 < 1 \iff |x| + |y| < 1$

$||x||_\infty = \max\{|x|,|y|\} < 1 \iff |x| < 1$ and $|y| < 1. If$ $x,y > 0 \implies x < 1$ and $y < 1$

1.5
SOME DISTANCE FUNCTIONS

$x, y \in \mathbb{R}^2,\ x = (x_1, x_2),\ y = (y_1, y_2)$

RIVER DISTANCE FUNCTION:

$$d_R(x,y) = \begin{cases} 0 & if\, x = y \\ |x_2| + |y_2| + |x_1 - y_1| & if\, x \neq y \end{cases}$$

Railway Distance Function:

$$d_r(x,y) = \begin{cases} 0 & if\, x = y \\ \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2} & if\, x \neq y \end{cases}$$

Note: The square root inside the Railway distance function is optional, and merely written for more accuracy.

Note on Schwartz Inequality:

Schwartz inequality is given in the form: $< x, y > \leq < x, x > . < y, y >$
$\frac{1}{p} + \frac{1}{q} = 1$
$\sum_{i=1}^{n} x_i\, y_i \leq \left(\sum_{i=n}^{n} x_i^p\right)^{1/p} . \left(\sum_{i=n}^{n} y_i^q\right)^{1/q}$
$< x, y > \leq ||x||_p . ||y||_q$

1.6 Limits

(X,b) is a metric space. We say that $x \in X$ is the limit of a sequence $\{x_n\}, \{x_1, x_2, x_3, ...\}$ if:

$$\lim_{n \to \infty} d(x_1, x_n) = 0$$

then we write $x_n \longrightarrow x$ as $n \longrightarrow \infty$.
the limit provided here can be interpreted this way: $\forall_{\epsilon > 0} \exists_N \forall_{n > N} d(x_1, x_n) < \epsilon$.

Proposition:

If $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and $x_n \longrightarrow y$ as $n \longrightarrow \infty$ then x = y.

Lemma:

If $x \neq y$ then there exists $\epsilon_1, \epsilon_2 > 0$ such that $B(x, \epsilon_1) \cap B(y, \epsilon_2) = \phi$
Proof.
Let there be two different balls with centers x and y and diameters $\epsilon_1\ and\ \epsilon_2$ respectively, and let $\epsilon_1 = \epsilon_2$.
Let $\epsilon = \frac{d(x,y)}{3}$. Suppose that $z \in B(x, \epsilon_1) \cap B(y, \epsilon_2)$ then $d(z, x) < \epsilon$. But then, *from triangle inequality:*
$d(x, y) \leq d(x, z) + d(z, y) < 2\epsilon = \frac{2}{3} d(x, y)$. So d(x,y) = 0, and then x = y.

Suppose that $x \neq y$ and $x_n \to x$ let $\epsilon = \frac{1}{3}d(x,y) > 0$ then: $\exists_N \forall_{n>N}\, d(x_n, x) < \frac{1}{3}d(x,y)$ and $x_n \to y$.

But then $\exists_N \forall_{n>N}\, x_n \in B(x, \frac{1}{3}d(x,y)) \cap B(x, \frac{1}{3}d(x,y))$ which is **impossible** for $x \neq y$

Q.E.D.

1.7

Some Examples of Matrices

1. $X = \mathbb{R}$

   d(x,y) = |x-y|
2. X is any set.

$$d(x,y) = \begin{cases} 0 & if\, x = y \\ 1 & if\, x \neq y \end{cases}$$

   It is called **Discrete Metric**. Let $x, y, z \in X$ if $x \neq y$ then either $z \neq x$ or $z \neq y$, so by triangle inequality:

   $d(x,y) = 1 \leq d(x,z) + d(y,z)$.

   If x = y; $d(x,y) = 0 \leq d(x,z)+d(z,y)$, but $d(x,z)+d(z,y) \geq 0$ because $d : X \times X \to [0, +\infty)$
3. Let (X, d_x) and (Y, d_y) be a metric spaces, then:

   $d_{x\times y}((x_1, y_1), (x_2, y_2)) = \sqrt{(d_x(x_1, x_2))^2 + (d_y(y_1, y_2))^2}$ is a metrix on $X \to Y$ **Cartesian**
4. Let (X,d) be a metric space. Then:

   $d_1(x,y) = min\{1, d(x,y)\} \leq 1$ always is a metric on X

   - $d_1(x,y) = 0 \iff d(x,y)0 \iff x = y$
   - $d_1(x,y) = d_1(y,x)$

Elaboration:

Let $x \neq y$, consider $z \in X$:

1.
   - $d(z,x) \geq 1 \quad \text{and} \quad d(z,y) \geq 1$
   - $d(z,x) \geq 1 \quad \text{and} \quad d(z,y) < 1$

   $d_1(y,z) = d_1(x,z) = 1$, but $d_1(x,y) \leq 1 \leq z = d_1(x,z) + d_1(y,z)$
2.
   - $d(z,x) > 1 \quad \text{and} \quad d(z,y) \leq 1$
   - $d(z,x) \geq 1 \quad \text{and} \quad d(z,y) \leq 1$

   $d_1(x,y) \leq 1 \leq 1 + d_1(z,y) = d_1(z,x) + d_1(z,y)$
3.
   - $d(z,x) < 1 \quad \text{and} \quad d(z,y) < 1$

     $d_1(x,y) \geq 1,\ d(z,x) < 1\ d(z,y) < 1$

     $d_1(x,y) = 1 \leq d(x,y) \leq d(x,z) + d(y,z) = d_1(x,z) + d_1(y,z)$
   - $d_1(x,y) = d(x,y)$

     $d_1(x,y) = d(x,y) \quad , d_1(x,z)d(x,z) \quad$, and $d_1(y,z) = d(y,z)$

Notes:

- In topology we don't care about big values, small numbers and neighbourhoods are important.
- The main goal of the previous examples is to show how to produce metric for a metric.

II

Sets

2.1 OPEN SETS

We say that a set $A \subset X$ is **open** if: $\forall_{x \in A} \exists_{r_x > 0} B(x, r_x) \subset A$.

In $(\mathbb{R}, ||.||) \equiv d(x,y) = |x - y|$ (0,1] is not open because
$1 \in (o, 1]$, but $\forall_{\epsilon > x} B(1, \epsilon) = (1 - \epsilon, 1 + \epsilon) \not\subset (0, 1]$ on the other hand (0,1) is open.
A set G is not open, if: $\exists_{x \in G} \forall_{r > 0} \exists_{z \in B(x,r)} z \notin G$

2.2 CLOSED SETS

We say that a set $C \subset X$ is **closed** if $X \setminus C$ is open.
$\forall_{x \in X \setminus C} \exists_{r_x > 0} B(x, r_x) \subset X \setminus C$.
$X \setminus C = \{x \in X : x \notin C\}$
A set $F \subset X$ is not closed if $X \setminus F$ is not open, if:
$\exists_{x \in X \setminus F} \forall_{r > 0} \exists_{z \in B(x,r)} z \notin X \setminus F \implies \exists_{x \in X \setminus F} \forall_{r > 0} \exists_{z \in B(x,r)} z \in F$.

ALTERNATIVE DEFINITION:

Let (X,d) be a metric space and let $F \subset X$. We say that the set F is closed if $X \setminus F$ is an open set. The following statement holds: F is closed if and only if all sequences x_n of points of the set F such that $x_n \to x$ for some $x \in X$, we have $x \in F$. In other words, the set F is closed if and only if every convergent sequence of points of F is convergent to a point of F.

2.3 Elaboration on Sets

Take a set C is closed, and $x \in X$ if and only if for every sequence $\{x_n\} \subset C$ such that $x_n \to x$. C is closed the sequence converges.

If C is not closed then: $\exists_{x \in X \setminus C} \forall_{\frac{1}{n}>0} \exists_{z_n \in B(x,\frac{1}{n})} z_n \in C \,; r = \frac{1}{n}$. **(*)**

Indeed, $\forall_{\epsilon>0} \exists_N \forall_{n>N} d(x, z_n) < \epsilon$ because $\forall_{\epsilon>0}$ take $\frac{1}{N} < \epsilon$ then from (*)

$z_n \in B(x, \frac{1}{n})$ for all n

$z_n \in B(x, \epsilon)$ for all n > N

$z_1 \in B(x, 1)$

$z_2 \in B(x, \frac{1}{2})$

$z_n \in B(x, \frac{1}{n}) \quad \forall_{n>N} z_n \in B(x, \frac{1}{N})$

Then let X be closed.

Let $x_n \to x \in X$.

Suppose that $x \notin X$, we will show that it is impossible.

The $x \in X \setminus C$ but $X \setminus C$ is open so $\exists_{r_x>0} B(x, r_x) \subset X \setminus C$. But $x_n \to x$ means that:

$\exists_N \forall_{n>N} d(x_n, x) < r_x$, so e.g., $x_{C \ni N+1} \in B(x, r_x) \subset X \setminus C$ which is a contradiction.

Notes:

- Let $\{x_n\} \subset C, x_n \to x : \forall_{\epsilon>0} \exists_N \forall_{n>N} d(x_n, x) < \epsilon \,, x_n \in B(x, \epsilon)$
- If the set is not closed then the limit is not in my set.

2.4 Continuity

Let (X, d), (Y, ρ) be metric spaces. We say that a function $f : (X, d) \to (Y, \rho)$ is continuous at $x. \in X$ if:

$\forall_{\epsilon>0} \exists_{\delta>0} d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \epsilon.$ **(*)**

Then, $f : (X, d) \to (Y, \rho)$ is continuous at $x_0 \in X$ if and only if $x_n \to x_0 \implies f(x_n) \to f(x_0)$

Definition:

$f : (X, d) \to (Y, \rho)$ is continuous if it is continuous at every $x \in X$.

Proof.

$\implies x_n \to x_0 \forall_{\epsilon_1>0} \exists_{N_1} \forall_{n>N_1} d(x_n, x_0) < \epsilon_1$ **(1)**.

We want to show that $\forall_{\epsilon>0} \exists_N \forall_{n>N} \rho(f(x_n), f(x_0)) < \epsilon$.

Let $\epsilon > 0$ then $\delta > 0$ such that (*) holds. Then for this δ from **(1)** we let $\exists_{N_1} \forall_{n>N_1} d(x_n, x_0) < \delta$ so $\forall_{n>N_1}$ from (*) $\rho(f(x_n), f(x_0)) < \epsilon$.

Because we can do all the above for any $\epsilon > 0$, we get $\forall_{\epsilon>0} \exists_{N_1} \forall_{n>N_1} \rho(f(x_n), f(x_0)) < \epsilon$.

$\Leftarrow$ Suppose that (*) doesn't hold; i.e., $\exists_{\epsilon>0} \forall_{\delta>0} \exists_{x_\epsilon} d(x_\epsilon, x_0) < \delta$ and $\rho(f(x_\epsilon), f(x_0)) \geq \epsilon$.

We will find $x_n \to x_0$ such that $f(x_n) \nrightarrow f(x_0)$.

Take $\epsilon > 0$ as above for $\delta_n = \frac{1}{n}$. We get $x_n : d(x_n, x_0) < \frac{1}{n}$ and $\rho(f(x_n), f(x_0)) \geq \epsilon$. $\{x_n\}$ is a sequence, $x_n \to x$, but $f(x_n) \nrightarrow f(x_0)$ because $\rho(f(x_n), f(x_0)) \geq \epsilon$.

Q.E.D.

Note: it is easier to define an open set than a metric space. But open sets are not local, so we will need a neighbourhood.

> DEFINITION: *On Metric Spaces*

$f : (X, d) \to (Y, \rho)$ is continuous if it is continuous at every $x \in X$.

The function $f : (X, d) \to (Y, \rho)$ is continuous at x_0 if and only if for every open set U in (Y, ρ) such that $f(x_0) \in U$. And $f^{-1}(U)$ is an open set in (X,d) contains an open set W in (X,d) such that $x \in W$.

Proof.

Suppose that F is continuous at x_0. Let $f(x_0) \in U$, U - open set in (Y, ρ).

Then there is $B(f(x_0), \epsilon) \subset U$. From (*) we get $B(x_0, \delta)$ such that:

$f(B(x_0, \delta)) \subset B(f(x_0), \epsilon) \subset U$.

$W := B(x_0, \delta) \subset f^{-1}(B(f(x_0), \epsilon)) \subset f^{-1}(U)$.

Suppose that for every open set U in (Y, ρ) such that $f(x_0) \in U$. There exists an open set W in (X,d) such that $x_0 \in W$, and $f(W) \subset U$.

We will show that f is continuous at x_0

Let $\epsilon > 0$take $B(f(x_0), \epsilon) = U$. Then there exists open set W such that $x_0 \in W$ and $f(W) \subset U$. But since W is open and $x_0 \in W$. There is $B(x_0, d) \subset W$ for some $\delta > 0$. But then:

$f(B(x_0, \delta)) \subset f(W) \subset U = B(f(x_0), \epsilon)$.

—

Suppose that F is continuous at every $x_0 \in X$. We will show that for every U open in (Y, ρ), $f^{-1}(U)$ is open in (X,d).

Let U be open in (Y, ρ), let $f(x_0) \in U$. Otherwise $f^{-1}(U) = \phi$ is open. For some x_0 there is an open set W_{x_0}, $x \in W_{x_0}$, $W_{x_0} \in f^{-1}(U)$. $f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} W_x$ so open at whole of open set.

For every open U $f^{-1}(U)$ $is\, open \implies$ f is continuous everywhere.

Let $x_0 \in X$ take any open U such that $f(x_0) \in U$ then $f^{-1}(U)$ is open and $x_0 \in f^{-1}(U)$ because $f(x_0) \in U$ so, $\exists_{\delta>0}\, B(x_0,\delta) \subset f^{-1}(U)$, $f(B(x_0,\delta)) \subset U$ in particular we can take $U = B(f(x_0),\delta)$ to get **(*)**

Q.E.D.

Note related to the proof:

- $f^{-1}(U) \supset W \iff U \supset f(W)$
- Pre-image: for a set $U \subset Y$. $f^{-1}(U) \subset X \implies f^{-1}(U) = \{x \in X : f(x) \in U\}$
- The union of all sets in this proof gives us a pre-image.

Examples of Metric Spaces:

Let (X,d) , (Y,ρ) - metric spaces.

C(X,Y) - set of continuous functions $f : X \to Y$, then: $\sigma(f,g) = \sup_{x\in X} \rho(f(x),g(x))$ is a metric if:

- $diam(Y,\rho) < +\infty$ or
- (X, d) is compact. or
- Instead of C(X,Y) and consider B(X,Y) *a set of bounded functions.*

Note: The distance between two bounded sets is bounded.

2.5 Related Concepts

Bounded Sets:

Let (A,d) - metric space.

The diameter of the set A is: $diam(A,d) = \sup_{x,y\in A} d(x,y)$. If $diam(A,d) < +\infty$ then we say that A is bounded. $\iff \exists_{+\infty>M>0} \forall_{x,y\in A}\, d(x,y) < M$

Sup:

Let $B \subset \mathbb{R}$ the $\sup_{x\in B} x$ is either $+\infty$ or a number M such that:

- $\forall_{x\in B} x \le M$ Best possible number.
- $\forall_{\epsilon>0} \exists_{x\in B} x > M - \epsilon$ To cover itself.

$x \in (0,1) : \sup_{x\in(0,1)} x = 1$.

There is another concept called: **infimum** $inf_{x\in B}x$ which has the same but the opposite properties and logic of **sup**.

NOTE:

- A function $f : x \longrightarrow (Y, \rho)$ is bounded if $diam(Y, \rho) < +\infty$.
- A is bounded if $diam(A) < +\infty$, and $diam(A) = \sup_{x,y\in A} d(x, y)$ if $diam(A) < +\infty$, then:
  - $\forall_{x,y\in A} d(x, y) < diam(A)$.
  - $\forall_{\epsilon>0} \exists_{x_\epsilon,y_\epsilon\in A} d(x_\epsilon, y_\epsilon) > diam(A) - \epsilon$. Taking $\epsilon = \frac{1}{n}$ we get a sequence of pairs (x_n, y_n). $diam(A) - \frac{1}{n} < d(x_n, y_n) \leq diam(A)$

SOME PROPERTIES:

$\forall_{x\in X} \rho(f(x), g(x)) \leq \rho(f(x), h(x)) + \rho(h(x), g(x))$
$\sup_{x\in X} \rho(f(x), g(x)) \leq \sup_{x\in X}\Big(\rho(f(x), h(x)) + \rho(h(x), g(x))\Big)$
$\leq \sup_{x\in X} \rho(f(x), g(x)) + \sup_{x\in X} \rho(h(x), g(x))$
For any functions F, G and any set B:
$\sup_{x\in B}(F(x) + G(x)) \leq \sup_{x\in B} F(x) + \sup_{x\in B} G(x)$
$\forall_{x\in B}\ F(x) + G(x) \leq \sup_{x\in B} F(x) + \sup_{x\in B} G(x)$

NOTES:

- if μ is such that $\forall_{x\in B} x \leq \mu$ then $\sup_{x\in B} x \leq M$ which is a contradiction.
- f,g: $[0, 1] \longrightarrow \mathbb{R}$, continuous:
  1. $\sigma(f, g) = \max_{x\in[0,1]}|f(x).g(x)|$
  2. $\sigma(f, g) = \int_0^1 |f(x).g(x)|dz$

EXAMPLES:

- Let $x \in X$. A set $A \subset X$ is bounded if and only if there exists R > 0 such that $A \subset B(x, R)$.
- Let $A \subset X$. For all $x \in X$ the distance from x to A is: $dist(x, A) = inf_{y\in A} d(x, y)$

INTERIOR:

- Let $A \subset X$. We say that $x \in A$ is a point of interior of A if $\exists_{r>0} B(x, r) \subset A$. $Int(A) \subset A$.
- Int(A) is open for any, but Int(A) might be ϕ for some A, and ϕ is not open.
- $B(x, r) \subset Int(A)$

Closure:

- For a set $A \subset X$ we define the closure of A, denoted by $\overline{A}$ or Cl(A) or d(A), and the set of closure points of A where $x \in X$ is a closure point of A if: $\forall_{\epsilon>0} \exists_{y \in A} d(y,x) < \epsilon \iff \forall_{\epsilon>0} B(x,\epsilon) \cap A \neq \phi$.
- The closure of A is the set of all limits of all $\{x_n\}$ where $x_n \in A$.
- A is closed $\iff$ A = $\overline{A}$.
- A is open $\iff$ A = Int(A).

Cluster:

x is an accumulation point or a *cluster point* of A if: $\forall_{\epsilon>0} \exists_{x,y \in A} d(x,y) < \epsilon$. We write then $x \in A^d$.
Let $A \subset X$. We say that $x_0 \in X$ is a *cluster point* of A if there exists a sequence x_n such that: $(x_n) \subset A$, $\forall_{n \in \mathbb{N}} x_n \neq x_0$, and $x_n \to x_0$. We denote by A^d the set of all *cluster points* of A.
For a set $A \subset X$ we define its *closure* $\overline{A}$ by the formula: $\overline{A} = A \cup A^d$

Density:

A set $A \subset X$ is **Dense** in X if $\overline{A} = X$ because everything is limit.
A set A in X is **Nowhere Dense** if $Int(\overline{A}) = \phi$

Notes:

- $X \setminus Int(\overline{A}) = x \iff Cl(x \setminus \overline{A} = x)$
- For ϕ we cannot find a ball so something from beyond the set, so a hole in every ball.
- $X \setminus Int(A) = Cl(X \setminus A)$
- $Int(X \setminus A) = X \setminus Cl(A)$
- $A = X \setminus (X \setminus A)$

Example:

Let (X,d) - metric space.
$y \in X$. Prove that the function f(x) = d(x,y) is continuous. $f : (X,d) \to (\mathbb{R}.|.|)$.
Solution:
f is continuous if $\forall_{\epsilon 0} \exists_{\partial >} d(x,x_0) < \partial \implies |f(x) - f(x_0)| < \epsilon$
$|f(x) - f(x_0)| = |d(x,y) - d(x_0,y)|$
$d(x,y) \leq d(x,x_0) + d(x_0,y)$

$$d(x_0, y) \leq d(x_0, x) + d(x, y) \begin{cases} d(x, y) - d(x_0, y) \leq d(x, x_0) & \textbf{(1)} \\ d(x_0, y) - d(x, y) \leq d(x, x_0) & \textbf{(2)} \end{cases}$$

$-(d(x, y) - d(x_0, y)) \leq d(x, x_0)$ **(3)**

$-d(x, x_0) \leq d(x, y) - d(x_0, y)$ **(4)**

From **(1)** and **(4)** $\forall_{x_0, x \in X} |d(x, y) - d(x, x_0) \leq d(x, x_0)$ **(*)**.

Let $\epsilon > 0$. We set $\epsilon = \delta$. Then if $d(x, x_0) < \delta$ from **(*)** we get $|f(x) - f(x_0)| \leq d(x, x_0) < \delta = \epsilon$.

Another way:

Show that if $x_n \longrightarrow x$ then $f(x_n) \rightarrow f(x)$.

Let $x_n \longrightarrow x$. Then $d(x_n, x) \longrightarrow 0$. $f(x_n) - f(x_0)| \leq d(x_n, x_0) \longrightarrow 0$ so $f(x_n) \longrightarrow f(x_0)$.

2.6 Compactness

Compact Set:

A set $A \subset X$ is compact if for every family of open sets U_α in X such that $A \subset \bigcup_\alpha U_\alpha$ *cover it* there exists a finite subfamily $\{U_{\alpha_1}, ..., U_{\alpha_N}\}$ such that $A \subset \bigcup_{i=1}^{N} U_{\alpha_i}$.

Example: (0,1) is not compact, such that it can be written in the form: $\bigcup_{n=1}^{\infty}(\frac{1}{n}, 1 - \frac{1}{n}) = (0, 1)$

Sequentially Compact Sets:

(X,d) is a metric space.

A set $A \subset X$ is compact, more prefferncy *sequentially compact* if for every sequence $\{x_n\} \subset A$ there is a suequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow x$ for $x \in A$ as $k \longrightarrow \infty$.

Lemma:

A compact set A is bounded and closed.

Proof.

We will prove that A is closed.

Let $\{x_n\} \subset A$, $x_n \longrightarrow x$.

We will show that $x \in A$. But from the definition of *Sequentially Compact Sets* there is $x_{n_k} \longrightarrow x_\infty$. So $x \in A$. *compactness means convergence.*

A is bounded $\iff$ $A \subset B(y, R)$, for some $y \in X, R > 0$.

By contradiction:

Suppose that A is compact and not bounded.

Let $y \in X$, take $n \in \mathbb{N}$ then $\forall_{n\in\mathbb{N}}\ \exists_{x_n\in A}\, x_n \in B(y, d(y, x_{n+1}) + 1)$.

Let $x_0 \in A$. x_n has k convergent subsequence.

If $x_{n_k} \longrightarrow x$, then $\forall_{\epsilon>0}\,\exists_k\,\forall_{n>k}\, d(x_{n_k}, x) < \epsilon$. In particular, $\forall_{\epsilon>0}\,\exists_k\, d(x_{n_k}, x) < \epsilon$ and $d(x_{n_k+1}, x) < \epsilon$.

$d(x_{nk}, x_{n_k+1} \leq d(x_{n_k}, x) + d(x, x_{n_k+1}) < 2\epsilon$.

So, if $x_n \longrightarrow x$ then $\forall_{\epsilon>0}\,\exists_k\, d(x_{n_k}, x_{n_k+1}) < 2\epsilon$.

But in the sequence $\{x_n\}$ we have $d(x_{n+1}, x_n) geq 1 \implies \forall_{n,m}\, d(x_n, x_m) \geq |n - m| \geq 1$.

$x_n \in \overline{B}(x, d(x_n, y))$. $x_{n+1} \notin B(y, d(y, x_n) + 1)$.

$d(x_{n+1}, y) > d(x_n, y)$.

$d(x_{n+1}, y) > d(y, x_n) + 1$.

$d(y, x_{n+1}) \leq d(y, x_n) + d(x_{n+1}, x_n) \implies d(x_{n+1}, x) \geq d(y, x_{n+1}) - d(y, x_n) > \frac{1}{2}\, or\ \geq 1$. Because $d(y, x_{n+1}) \geq d(y, x_n)$; $_{n+1} \notin \overline{B}(y, d(y, x_n) + 1)$.

Q.E.D.

Another proof that a compact set A is bounded:

For ant $y \in X$. $X = \bigcup_{n=1}^{\infty} B(y, n)$. So in particular, $A \subset \bigcup_{n=1}^{\infty} B(y, n)$. From definition of *Compact Sets*, $A \subset \bigcup_{k=1}^{N} B(y, n_k)$ for some $\{n_k\}$, But then $A \subset B(y, \max\{n_k\})$.

Q.E.D.

THEOREM:

If A is compact then it is sequentially compact.

Proof.

Let $\{x_n\} \subset A$ be a sequence. Suppose that $\{x_n\}$ has k convergent subsequence.

Then $\forall_{x\in A}\,\exists_{\delta(x)>0}\,\exists_{N(x)}\,\forall_{n\geq N(x)}\, x_n \notin B(x, \delta(x))$.

$A = \bigcup_{x\in A}\{x\} \subset \bigcup_{x\in X} B(x, \delta(x))$.

From the definition of *Compact Sets*: $A = \bigcup_{\delta=1}^{M} B(y_i, \delta(y_j))$.

set $\mathcal{N} = \max_{1\leq j\leq M} N(y_j)$. Since $\mathcal{N} \geq N(y_j)$, $\forall_{1\leq j\leq M}\, x_{\mathcal{N}} \notin B(y_j, \delta(y_j))$.

So $x_{\mathcal{N}} \notin \bigcup_{j=1}^{M} B(y_j, \delta(y_j)) \supset A$ so $x_{\mathcal{N}} \notin A$ which is a contrdiction.

Q.E.D.

LEMMA:

Suppose that A is sequentially compact, and a set A is covered by $\bigcup_\alpha U_\alpha$ for some open sets $\{U_\alpha\}$. Then there exists $\delta > 0$ such that for every $x \in A$ there exists $\alpha(x)$ such that $B(x,\delta) \subset U_{\alpha(x)}$. δ is called *Lebesgue's number* of the cover $\{U_\alpha\}$.

Proof.

The contradiction of $\exists_{\delta>0} \forall_{x\in A\ \alpha(x)} B(x,\delta) \subset U_{\alpha(x)}$ is: $\forall_{\frac{1}{n}=\delta>0} \exists_{x_n} \forall_\alpha B(x,\frac{1}{n}) \not\subset U_\alpha$.

Suppose that $A \subset \bigcup_\alpha U_\alpha$, but there is no $\delta > 0$ such that $\forall_{x\in A} B(x,\delta) \subset U_{\alpha(x)}$. Then for each $n \geq 1$ we can find x_n such that $\forall_\alpha B(x,\frac{1}{n}) \not\subset U_\alpha$. **(*)**.

From *Sequential Compactness*, there is a subsequence $x_{n_y} \to y \in A$.

Since $y \in A$, there is U_β such that $y \in U_\beta$. Since U_β is open, $\exists_{\epsilon>0} B(y,\epsilon) \subset U_\beta$.

Now let J be such that $n_J > \frac{2}{\epsilon}$ and $d(x_{n_J}, y) < \frac{\epsilon}{2}$ *using triangle inequality*.

$B(x_{n_J}, \frac{1}{n_J}) \subset B(x_{n_J}, \frac{2}{\epsilon} \subset B(x,\epsilon)$, which contradicts**(*)**.

$x \in B(x_{n_J}, \frac{\epsilon}{2}) \implies d(x, x_{n_J}) < \frac{\epsilon}{2}$.

But then: $d(y,x) \leq d(x,x_{n_J}) + d(x_{n_J}, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Q.E.D.

LEMMA:

(X,d), (Y,ρ) are metric spaces.

If K is a compact set, $K \subset X$, $f : X \to Y$ is continuous then f(K) is compact.

Proof.

Let $\{U_\alpha\}$ be a family of open sets, such that $f(K) \subset \bigcup_\alpha U_\alpha$. Then $f^{-1}(U_\alpha)$ are open sets such that $K \subset \bigcup_\alpha f^{-1}(U_\alpha)$. From *compactness* of k: there is a finite subfamily $\{f^{-1}(U_{\alpha_j})\}_{j=1}^N$.

$f(K) \subset f(\bigcup_{j=1}^N f^{-1}(U_{\alpha_j})) \subseteq \bigcup_{j=1}^N f(f^{-1}(U_{\alpha_j})) \subset \bigcup_{j=1}^N U_{\alpha_j}$

Q.E.D.

Note: $f(A \cap B) \neq f(A) \cap f(B)$

COROLLARY:

If K - compact set.

f - contentious. $f : K \to \mathbb{R}$

Then f attain $min_k f$ and $\max_k f$, i.e., $\exists_{x_1,x_2\in K} f(x_1) = min_{x\in K} f(x)$, $f(x_2) = \max_{x\in K} f(x)$.

f(K) is compact, so it is bounded and closed.

$f(k) \subset [a,b]\ so\ \exists_M \forall_{x\in K} f(x) \leq M$; $\sup_{x\in K} f(x) \leq M$.

Proof.

$f(K) \subset \mathbb{R}$ is compact so $\sup_{x \in k} f(x) \leq M$ so $\forall_n \exists_{x_n \in K} M \geq f(x_n) \geq M - \frac{1}{n}$.

$f(x_n)$ is increasing for $n \in \mathbb{R}$ but f(x) is closed so $a \in f(x)$ so $f(x_n) \to f(x) = \max_{x \in K} f(x)$.

sup can be used as well.

Q.E.D.

Another proof. K is *sequentially compact*. $\forall_n \exists_{x_n} \sup_{x \in K} f(x) - \frac{1}{n} \leq f(x_n) n \leq \sup_{x \in K} f(x)$.

$\{x_{n_k}\}$ subsequence of $\{x_n\}$; $x_{n_k} \to x \in$; $f(x_{n_k}) \to f(x)$ because of continuity.

Q.E.D.

EXAMPLE:

Let $a, b \in \mathbb{R}$.

$[a, b]$ is compact in $\mathbb{R}$ with usual distance $d(x, y) = |x - y|$ **Bolzano–Weierstrass theorem.**

Proof.

$\{x_n\} \subset [a, b]$. Let $a_0 = a$, $b_0 = b$. For $n \geq 1$ we devide $[a_0, b_0]$ in half obtaining $[a_n, b_n]$ in which there is infinitly many elements of $\{x_n\}$.

For any $k \in \mathbb{N}$ pick any $x_n \in [a_k, b_k]$.

$\{x_n\}$ is a sequence of $\{x_{n_k}\}$ is a sequence of $\{x_n\}$ and we have $b \geq a_k$ is increasing while $a \leq b_k$ is decreasing $\implies$ $a_k \leq x_{n_k} \leq b_k$ $\implies$ convergent. Or we can write it this way: $\bar{a} \leq x_{n_k} \leq \bar{b}$, and $\bar{a} = \bar{b}$ because $|b_k - a_k| = \frac{1}{2^k}$ so $x_{n_k} \to x \in [0, 1]$.

Q.E.D.

LEMMA:

If K is a compact set, $C \subset K$, and C is closed then C is conpact.

Proof.

$\{x_n\} \subset C$ a sequence such that $x_{n_k} \to x \in K$, but since $\{x_{n_k}\} \subset C$, we have $x \in C$.

Q.E.D.

LEMMA:

Let K be a compact set. For every $\epsilon > 0$ there is $N \in \mathbb{N}$ and $\{x_1, ..., x_N\} \subset K$ such that $K \subset \bigcup_{i=1}^{N} B(x_i, \epsilon)$.

Proof.

Take an open cover $K \subset \bigcup_{x \in K} B(x, \epsilon)$ and find a finite subfamily $\{B(x_i, \epsilon), x_i \in \{1, ..., N\}\}$.

Q.E.D

NOTES:

- The image of a compact set is compact.
- Every closed ad bounded set in $\mathbb{R}^n$ with Euclidean metric or Equivalent to it is compact.

2.7

COMPLETENESS

DEFINITION:

A sequence $\{x_n\}$ in a metric space (X,d) is a *Cauchy Sequence* if:

$\forall_{\epsilon>0} \exists_{N \in \mathbb{N}} \forall_{n,m>N} \; d(x_n, x_m) < \epsilon$, equivalently: $\forall_{\epsilon>0} \exists_{N \in \mathbb{N}} \forall_{n>0} \; d(x_n, x_m) < \epsilon$.

$(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m) < \epsilon + \epsilon < 2\epsilon$.

COMPLETE SET:

A metric space is complete if every Cauchy sequence in this space is convergent; *has a limit which belongs to this space.*

Example:

$\mathbb{R}$ is complete with $d(x, y) = |x - y|$, but $\mathbb{R} \supset \mathbb{Q}$ is not complete $q_m \to \sqrt{2}$, $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof.

If $\{x_n\}$ is a Cauchy sequence: $\forall_{\epsilon>0} \exists_N \forall_{n>N} \; d(x_n, x_N) < \epsilon$ **(*)**

$\iff \forall_{n>N} \; x_n \in B(x_N, \epsilon) \subset \overline{B}(x_N, \epsilon)$, which is closed, and a ball is bounded set. In $\mathbb{R}$ it is acutally $[x_N - \epsilon, x_N + \epsilon]$ *Bolzano–Weierstrass theorem* $\implies$ $[x_N - \epsilon, x_N + \epsilon]$ is compact $\supset \{x_n\}_{n>N}$ so there is a subsequence $\{x_{n_k}\}$, $x_{n_k} \to x_\infty \in [x_N - \epsilon, x_N + \epsilon] \subset \mathbb{R}$. If a Cauchy sequence $\{x_n\}$ has a convergent subsequence $\{xn_k\}, x_{n_k} \to x_\infty$ then $x_n \to x_\infty$.

$\forall_{\epsilon>0} \exists_N \forall_{n,m} d(x_n, x_m) < \epsilon$, $x_{n_k} \to x_\infty \implies \forall_{\epsilon>0} \exists_K \forall_{k \in K} \; d(x_{n_k}, x_\infty) < \epsilon$, let $\epsilon > 0$:

Let $N_1 : \forall_{n \geq N_1} \; d(x_n, x_{N_1}) \leq \frac{\epsilon}{2}$. **(1)**

Let $N_2 : \forall_{k \geq N_2} d(x_{n_k}, x_\infty) \leq \frac{\epsilon}{2}$. **(2)**

Then, $N = \max\{N_1, n_{N_2}\}$; $\forall_{n>m} \; d(x_\infty, x_m) \leq d(x_\infty, x_N) + d(x_N, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ so x_∞ is the limit of $\{x_n\}$.

From (*) every Cauchy sequence $\{x_n\}$ is bounded because $\forall_{\epsilon>0} \exists_N \{x_n\} = \{x_1, ..., x_N\} \subset \{x_{N+1}, ...\}$ and $\{x_{N+1}, ...\} \subset B(x_N, \epsilon)$. $\{x_n\}_{n=1}^{\infty} \subset B(x_n, \max\{\epsilon, \max_{J=\{1,...,N\}} d(x_N, x_J) + 1\})$.

Q.E.D.

BOUNDED CONTINUITY:

A function $f : (X, d) \to (Y, \rho)$ is bounded if:
$\exists_{Y \in y_f, R_f > 0} \forall_{x_i n X} f(x) \in B(y_f, R_f)$.
For f,g bounded functions, $\sigma(f, g) < +\infty$ because:
$\forall_{x \in X} \rho(f(x), g(x)) \leq \rho(f(x), y_f) + \rho(y_f, y_g) + \rho(g(x), y_g) < R_f + d(y_f, y_g) + R_g$.
Example:
$f(x) = x^n, f_n(x) \to 0 as n \to \infty$, $\forall_{x \in [0,1)}, f_n(1) = 1$ so it does not go uniformly to zero.

THEOREM:

Let (X,d) be a compact metric space; let (Y, ρ) be a complete metric space. Then the space $\left(C(X, Y), \sigma\right)$ of bounded continuous functions from (X, d) to (Y, ρ) with the distance function:
$\sigma(f, g) = \sup_{x \in X}(f(x), g(x))$ is complete.
Proof.
Let $\{f_n\}$ be a Cauchy sequence in $\left(\mathbb{C}(X, Y), \sigma\right)$.
$\forall_{\epsilon>0} \exists_N \forall_{n,m>N} \sigma(f_m, f_n) < \epsilon \implies \sup_{x \in X} \rho(f_n(x), f_m(x)) < \epsilon$, and $\sigma(f_n, f_\infty) \to 0$ because:
$\forall_{\epsilon>0} \exists_N \forall_{n,m>N} \forall_{x \in X} \rho(f_n(x), f_m(x)) < \epsilon, m \to \infty \implies \forall_{\epsilon>0} \exists_N \forall_{n>N} \forall_{x \in X} \rho(f_n(x), f_\infty(x)) < \epsilon$.
$\forall_{\epsilon>0} \exists_N \forall_{n,m>N} \forall_{x \in X} \rho(f_n(x), f_m(x)) < \epsilon$, so $\forall_{x \in X} \{f_n(x)\}$ is a Cauchy sequence, since (Y, ρ) is complete, $\forall_{x \in X} f_n(x) \to f_\infty(x)$, we can define a function $f_\infty(x)$ such that $f_\infty(x) = f_\infty(x)$, we will show that $f_\infty \in \mathbb{C}(X, Y)$, is continuous that: $\forall_{\epsilon>0} \exists_{\delta>0} d(x, y) < \delta \implies \rho(f_\infty(x), f_\infty(y)) < \epsilon$.
Let $\epsilon > 0 \exists_N \forall_{n,m>N} \sigma(f_n, f_m) < \frac{\epsilon}{9}$.
Let $\delta > 0$ such that $d(x, y) < \delta \implies \rho(f_N(x), f_N(y)) < \frac{\epsilon}{9}$.
Let n> N be such that $\rho(f_\infty(x), f_n(x)) < \frac{\epsilon}{3}$, and $\rho(f_\infty(y), f_n(y)) < \frac{\epsilon}{3}$, considering that N is big enough we have:
$\rho(f_\infty(x), f_\infty(y)) \leq \rho(f_\infty(x), f_n(x)) + \rho(f_n(x), f_N(x)) + \rho(f_N(x), f_N(y)) + \rho(f_N(y), f_n(y)) + \rho(f_n(y), f_\infty(y))$
$< \frac{\epsilon}{3} + \frac{\epsilon}{9} + \frac{\epsilon}{9} + \frac{\epsilon}{9} + \frac{\epsilon}{3} = \epsilon$, and this shows the completeness.

Q.E.D.

Note: Compactness is stronger than completeness, because a compact set is v=bounded and $f_y(x) = d(y, x)$ and compact $\implies$ continuous, but for a complete set it is not necessarily continuous, and not always bounded like $\mathbb{R}$ unless the balls have a cover $\implies$ bounded $\implies$ complete, and we need a boundary.

LEMMA:

(X,d) is a complete metric space.

A compact metric space, i.e., every sequence $\{x_n\}$ has a convergent subsequence.

Proof.

Let $\{x_n\}$ be a Cauchy sequence. Then in compact space there is a sequence $x_{n_k} \to x_\infty \in X$ but then $x_n \to x_\infty$, from this step you can look to a similar proof mentioned before in the previous topics, which is related to that $\mathbb{R}$ is complete, and the closed interval [ ] in $\mathbb{R}$ is compact. For more details think if there is something is locally compact...

EXERCISE:

If for every ball B(x,r) in (X,d) there exists a compact set $\mathbb{C}_{B(x,r)} \subset X$ such that $B(x,r) \subset \mathbb{X}_{B(x,r)}$ then (X,d) is complete.

$(x_1, x_2) \in \mathbb{R}$

$(x_1, x_2, x_3) \in \mathbb{R}^3$

$(x_1, x_2, x_3, ...) \in \mathbb{R}^\infty$ - sequence

$(1, 2, ...) \notin \mathcal{E}^\infty$ - not bounded

Solution:

There are spaces where $\overline{B}(x, r)$ is not compact, for example: $\mathcal{E}^\infty$ is a space of sequences $\{x_n\}$, $\forall_{x_n} x_n \in \mathbb{R}$ such that $\sup_{n\in\mathbb{N}}|x_n| < +\infty$ bounded, *not big*, with the distance function: $d(\{x_n\}, \{y_n\}) = \sup_{n\in N}|x_n - y_n|$, then take:

$\{x^{(1)}\} = (1, 0, ...)$

$\{x^{(2)}\} = (0, 1, 0, ...)$

$\{x^{(3)}\} = (0, 0, 1, 0, ...)$

Then $\{x^{(m)}\}$ has no convergent subsequence because $\forall_{n\neq m}\, d(\{x^{(n)}\, \{x^{(m)}\}) = 1$, but actually $\mathcal{E}^\infty$ is complete, *see proof of completeness of $C(X, \mathbb{R}) \cap B(x, R)$*, continuous and bounded real valued functions.

Notes on the proof.

- Set of continuous functions from $X \to Y$,
- This is a complete space with distance function $\rho(f,g) = \sup_{x\in X} d_Y(f(x), g(x))$.
- Set of bounded functions from $X \to Y$.

CONTRACTION:

Let (X,d) be a metric space, a function $f : X \to X$ is called a contraction if there exists M > 1 such that $\forall_{x,y\in X} d(f(x), f(y)) \leq M.d(x,y)$ **(*)**.

We will write: $f^2(x) = f(f(x))$

$f^n(x) = f^{n-1}(f(x))$ etc.

The main question is as we go more and more in functions does that implies the following:

$d(f^n(x), f^n(y)) \to 0$? *In fact there is an entire theorem Related to it.*

$d(f(x), f^2(x)) < d(f^2(x), f^3(x))$.

BANACH CONTRACTION PRINCIPLE:

Let (X,d) be a metric space.

Let $f : X \to X$ be a contraction.

Then there exists a unique $x_0 \in X$ such that:

- $\forall_{x\in X} \lim_{n\to\infty} f^n(x) = x_0$.
- $f(x_0) = x_0$, as the contraction goes more and more x_0 becomes the fixed point of f.

Proof.

Let $x \in X$, we will show that $\{f^n(x)\}$ is a Cauchy sequence.

Let n > m, $d(f^n(x), ff^n(x)) = d(f^{n-m}(f^m(x)), f^n(x))$.

Let's look at $d(f^k(x), x) \leq d(f(x), x) + d(f(f(x)), f(x)) + f(f^2(x)), f^2(x)) + ... + d(f(f^{k-1}(x)), f^{k-1}(x))$.

$d(f^k(x), x) \leq d(f(x), x) + d(f(f(x)), f(x)) + d(f^2(f(x)), f^2(x)) + d(f^3(f(x)), f^3(x)) + ... + d(f^{k-1}(f(x)), f^{k-1}(x))$ **(1)**

$d(f^2(x), f^2(y)) = d(f(f(x)), f(f(y))) \leq M.d(f(x), f(y)) \leq M^2.d(x,y)$

$\implies d(f^n(x), f^n(y)) \leq M^n.d(x,y)$ **(**)**

$d(f^n(x), f^n(y)) = d(f(f^{n-1}(x)), f(f^{n-1}(y))) \leq M.d(f^{n-1}(x), f^{n-1}(y)) \leq M.M^{n-1}d(x,y)$.

From (*), and by using **(**)**:

$d(f^k(x), x) \leq d(f(x), x) + M.d(f(x), x) + M^2.d(f(x), x) + ... + M^{k-1}d(f(x), x) =$

$= d(f(x), x).(1 + M + M^2 + ... + M^{k-1})$ **(***)**

From (***) and denoting: k = m-n, $x = f^k(x)$:
$d(f^n(x), f^m(x)) \leq d(f(f^m(x)), f^m(x)).(1 + M + M^2 + ... + M^{n-m-1}) =$
$= d(f^m(f(x)), f^m(x)).(1 + M + M^2 + ... + M^{n-m-1})$
By (**) $\leq M^m d(f(x), x).(1 + M + M^2 + ... + M^{n-m-1}) \leq d(f(x), x).M^m.\frac{1}{1-M}$.
Notice that: $\sum_{a=0}^{\infty} M^a = \frac{1}{1-M}$
$\forall_{n>m} d(f^n(x), f^m(x)) \leq d(f(x), x).M^m.\frac{1}{1-M}$ and as $m \to \infty$, $M < 1$ for every $\epsilon > 0$ we can take N such that $d(f(x), x).M^m.\frac{1}{1-M} < \epsilon$, then:
$\forall_{n,m>N} d(f^n(x), f^m(x)) \leq d(f(x), x).M^m.\frac{1}{1_M} \leq d(f(x), x).M^N.\frac{1}{1-M} < \epsilon$. so $\{f^n(x)\}$ is a Cauchy sequence.

Q.E.D.

LEMMA:

(X,d) is complete so $f^n(x) \to x_\infty \in X$, *notice that both sides does not depend on each other and* $f^n(x)$ *is a Cauchy sequence.* We will show that $f(x_\infty) = x_\infty$.
Proof.
From **(*)**, also we have that f is continuous, and so is f^n, so:
$f(x_\infty) = f(\lim_{n\to\infty} f^n(x))$ *by continuity* $= \lim_{n\to\infty} f(f^n(x)) = \lim_{n\to\infty} f^{n+1}(x) =$
$= \lim_{n\to\infty} f^n(x) = x_\infty$.
x_∞ is the unique fixed point of f, i.e., if f(x) = x and f(y) = y, then x = y.
If $x \neq y$, f(x) = x, f(y) y then $d(f(x), f(y)) \leq M.d(x, y) < d(x, y)$ which is **contradiction.** .

APPLICATION OF FIXED POINT:

We look for solutions of Ordinary Differential Equations (ODE):

$$\begin{cases} x'(t) \equiv \frac{dx(t)}{dt} = g(x(t)) & \text{g is even} \\ x(0) = x_0 & \text{initial condition} \end{cases}$$

Proof.
Since $x(t) = x(0) + \int_0^t x'(s)ds$, the solution would be $x(t) = x_0 + \int_0^t g(x(s))ds$.
If $x(t) = x_0 + \int_0^t g(x(s))ds$ and $x(0) = x_0$ then $x'(t) = g(x(t))$, and $f(x)(t) = x_0 + \int_0^t g(x(s))ds$
if f(x) = x then x is a solution of the ODE.
Consider a space $C_0([0, \epsilon], \mathbb{R})$ of continuous functions y on $[0, \epsilon]$ with values in $\mathbb{R}$ such that $y(0) = y_0$ with distance function: $\rho(x, y) = \max_{t\in[0,\epsilon]} |x(t) - y(t)|$

We have the following questions in mind:

- $[o, \epsilon]$ is compact, but is it complete?
- Can $f : x(t) \longrightarrow x_0 + \int_0^t g(x(s))ds$ be a contraction on the space $C_0([0, \epsilon], \mathbb{R})$?

$\rho(f(x), f(y)) = \max_{t\in[0,\epsilon]}|x_0 + \int_0^t g(x(s))ds - x_0 - \int_0^t g(y(s))ds| \leq$
$\leq \max_{t\in[0,\epsilon]}|\int_0^t (g(x(s)) - g(y(s)))ds| \leq \max_{t\in[0,\epsilon]} \int_0^t |g(x(s)) - g(y(s))|ds \leq$
$\leq \max_{t\in[0,\epsilon]} \int 0^t M.|x(s) - y(s)|ds \leq \int_0^\epsilon M.\rho(x, y)ds \leq M.\epsilon.\rho(x, y)$ and $M.\epsilon < 1$.
The last step was based on the following:
Suppose that $g \in C(\mathbb{R}, \mathbb{R})$ i.e., g' exists and is continuous, then:
$\forall_{a,b\in\mathbb{R}} \exists_{c\in[a,b]} |g(a) - g(b)| \leq g'(c).|a - b| \leq \max_{c\in[a,b]}|g'(c)|.|a - b|$.
Let us assume that there exists M > 0 sucht that: $\max_{x\in\mathbb{R}}|g'(c) \leq M$, e.g., M is **Lipschitz** and small enough.

Q.E.D.

Another Way:
$x(t) = x_0 + \int_0^t f(x(s))ds \implies$ goes always to this space, M, $C_{x_0}([-\epsilon, \epsilon], [-M + x_0, x_0 + M])$ while C_1 is local Lipschitz, C_{x_0} is a closed subset of all continuous functions, if it is complete
$\implies f(0) = x_0 = 0$.
$f([-M + x_0, x_0 + M]) \subset [-N_1, N_2]$
$f'(-M + x_0, x_0 + M) \subset [-N_2, N_3]$, which are both bounded on some interval.
$x(t) = x_0 + \int_0^t f(x(s))ds$ **(*)**.
$|x_0 + N_1.\epsilon| < |x_0 + M|$, $\epsilon < \frac{M}{N_1}$, and $\epsilon.N_2 < 1 \implies \epsilon < \frac{1}{N_1}$, to make sure that the map **(*)** is a contraction.
$|x(t) - y(t)| \leq \int_0^\epsilon |f(x(s)) - f(y(s))|ds \leq N_2.\epsilon|x(t) - y(t)| < 1 \implies$ contraction.
Note: $|x_0 + \int_0^t f(x(s))ds| \leq |t_0 + \epsilon N_1|$, and we can substitue $x_0 := 0$.

EXAMPLE:

$\mathbb{R} : x \longrightarrow \sqrt{x}.x \in [\delta, +\infty) \implies \sqrt{x} \in [\delta, +\infty)$, $\delta > 0$.
$|\sqrt{x} - \sqrt{y}|$ is there M $\implies |\sqrt{x} - \sqrt{y}| \leq M.|x - y|$?
$f(x) = \sqrt{x}, f'(x) = \frac{1}{2\sqrt{x}}$ if $\sqrt{x} \geq \sqrt{\epsilon} \implies f'(x) = \frac{1}{2\sqrt{x}} \leq \frac{1}{2\epsilon}$, so if $x \in [\delta, +\infty)$then $f^\infty(x) \leq \frac{1}{2\sqrt{\delta}}$, so $|\sqrt{x} - \sqrt{y}| \leq \frac{1}{2\sqrt{\delta}}|x - y|$, when $\frac{1}{2\sqrt{\delta}} < 1 \implies \sqrt{\delta} > \frac{1}{2} \implies \delta = \frac{1}{4} \implies \forall_{\epsilon>)}\, on\, [\frac{1}{4} + \epsilon, +\infty)$.
$\sqrt{1} = 1 \implies f(x) = \sqrt{x}$ is a contraction.
$\forall_{y\in[\frac{1}{4}+\epsilon,+\infty)} f^n(y) = y^{2n} \rightarrow 1\, as\, n \rightarrow \infty$, which is more general.

NOTES:

- The image of a compact set is compact, in other words, If f is continuous and X is compact then f(X) is compact, but for example let $f(x) = \arctan(x), f : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2}), \mathbb{R}$ is a complete space while $(-\frac{\pi}{2}, \frac{\pi}{2})$ is not complete because open interval is not complete.
- The image of a complete space is not necessarily complete.
- Every space can be completed.

CANTOR

A metric space is (X,d) is coomplete *every Cauchy sequence is complete* if and only if for every sequence of non-empty closed sets $C_i \subset X$ such that $\lim_{n\to\infty} diam(C_n) = 0$ and $\forall_{n\in\mathbb{N}} C_{n+1} \subset C_n$. The intersection $\bigcap_{n=1}^{\infty} C_n$ is non-empty. *Decreasing.*

$\Longrightarrow$

Suppose (X,d) is complete. Let $x_1 \in C_1$, $x_2 \in C_2$, ..., $x_n \in C_n$.

We will show that $\{x_n\}$ is a Cauchy sequence, i.e., $\forall_{\epsilon>0} \exists_N \forall_{n>N} d(x_n, x_N) < \epsilon$ **(1)**.

Let $\epsilon > 0$ from $\lim_{n\to\infty} diam(C_n) = 0$, then N is such that $\forall_{n\geq N} diam(C_n) \leq \epsilon$

$\Longrightarrow \forall_{n\geq N} d(x_n, x_N) < \epsilon$ as **(1)**, because $\forall_{n\geq N} x_n \in C_n \subset C_N$, because (X,d) is complete there is $x \in X$ such that $x_n \to x$.

We need to prove that $x \in \bigcap_{n=1}^{\infty} C_n$.

For every m we have $\forall_{n>m} x_n \in C_m$ so x is the limit of the sequence $\{x_n\}_{n=m+1}^{\infty}$ of elements of C_m.

Since C_m is closed and $x_n \to x$ we get $x \in C_m \implies x \in \bigcap_{n=1}^{\infty} C_n$.

We can prove also that $\bigcap_{n=1}^{\infty} C_n = \{x\}$, because if there is $y \neq x$, $y \in \bigcap_{n=1}^{\infty} C_n$ we have $0 < d(x, y)$, so we can take C_m such that $C_m < \frac{d(x,y)}{3}$. Then $x, y \in C_m$ so $d(x, y) \leq \frac{d(x,y)}{3}$ which is impossible for $0 < d(x, y)$.

$\Leftarrow$

Let $\{x_n\}$ be a Cauchy sequence. We will show that $\exists_{x\in X} x_n \to x$.

$C_1 = \overline{\{x_1, x_2, ...\}}$

$C_n = \overline{\{x_n, x_{n+1}\}}$

We will show that $diam(C_n) \to 0$

Let $\epsilon > 0$, $\exists_N \forall_{n,m>N} d(x_n, x_m) < \epsilon. diam(C_{N+1}) = diam\left(\overline{\{x_{N+1}, x_{N+2}, ...\}}\right) = diam(\{x_{N+1}, x_{N+2}, ...\}) = \sup_{a,b\in\{x_{N+1},x_{N+2},...\}}\left(d(a, b) \leq \epsilon\right). C_{n+1} \subset C_n \implies diam\left(C_{n+1}\right) \leq diam\left(C_n\right)$.

$\forall_{n>N} diam\left(C_n\right) \leq \epsilon , for\ \epsilon > 0 \implies x \in C_m \implies x \in \overline{\{x_m, x_{m+1}, ...\}}. \forall_{\epsilon>0}, \exists_{x_n}$ where $n > m , d(x, x_n) < \epsilon$ **(2)**.

Let $\epsilon > 0$, N such that $diam\left(C_n\right) < \frac{\epsilon}{3}$ and from **(2)** $\implies \exists_{x_m, M>N} d(x, x_M) < \frac{\epsilon}{3}$.

But then $\forall_{n>N} d(x_n, x) \leq d(x_n, x_M) + d(x_M, x) < \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon\ so\ x_n \rightarrow x$.

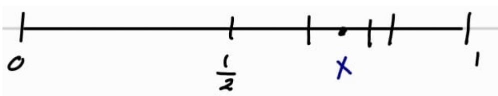
Q.E.D.

THEOREM:

A metric space is compact $\iff$ for every family of closed sets $\{C_n\}$ such that every finite subfamily $C_{n_1}, C_{n_2}, ..., C_{n_k}$ has non-empty intersection, we have $\bigcap_{n=1}^{\infty} = \phi$ *can be more than one point.*

EXAMPLE:

Let $C_1 = [0, 1]$
$C_1 \supset C_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$
$C_1 \supset C_2 \supset C_3 = [0, \frac{1}{4}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ (*)
$C_1 \supset C_2 \supset C_3 \supset ...$ which is closed. $C = \bigcap n = 1^{\infty} C_n$ being an intersection $\implies$ it is closed.
Constructing numbers from 0,1:



x $= 2.\frac{1}{3} + 2.\frac{1}{9} + \ldots$, $x = \sum_{n=1}^{\infty} b_n.\frac{1}{3^n}$, $b_n \in \{0, 1, 2\}$. **Rafering to (*):**
In the compact set we have $x = \sum_{n=1}^{\infty} b_n.\frac{1}{3^n} but b_n \in \{0, 2\}$ and 1 is removed because we remove the mid intervals. We can get $\sum a_n.\frac{1}{2^n}$ from it, but it is difficult in series language.
$\implies f$: cantor set $\rightarrow [0, 1]$
$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{b_n}{2}$ where $x = \sum_{n=0}^{\infty} b_n.\frac{1}{3^n}$.

LEMMA:

A cantor set C is a nowhere dense set. Int$(\overline{C}) = \phi$.
$x = \sum_{n=1}^{\infty} b_n.\frac{1}{3^n}$, $b_n \in \{0, 2\}$, $x = \lim_{N\rightarrow\infty} \sum_{i=1}^{N} b_i.\frac{1}{3^i}$
$y \in B(x, \epsilon)$, in particular $\forall_{\epsilon>0} \exists_N \sum_{i=1}^{N} b_i.\frac{1}{3^i} \in B(x, \epsilon)$, $b_i \in \{0, 2\}$.
Consider: $y = \sum_{i=1}^{N} b_i \frac{1}{3^i} + 1.\frac{1}{3^{N+M}}$ for elements from cantor set there is a neighbourhood $\notin C$ so the cantor set has holes inside it.
Notes:

- Intervals are like induction, and the question is *is there something left?*
- Hilbert space: $< x, y > \ \sum x_i y_i$,has a geometry while metric space does not. *projecting some elements.*

III

Topology

Before getting into this chapter it is worth mentioning that in metric spaces we follow such steps, *at least in a general sense*:

- Distance function, e.g., $d : X \to [0, +\infty)$
- Checking the conditions for the distance function.
- Constructing a ball, e.g., $B(x, r) = \{y \in X : d(y, x) < r\}$
- $U \subset X$ is open $\iff \forall_{x \in U} \exists_{r>0} B(x, r) \subset U$. *continuity.*
- Compactness.
- By a metric we can precede to the completeness.

In topology; however, we start straight by constructing the ball which makes it easier.

3.1
Topological Spaces

Definition:

A topology on X is a collection of sets *open sets* $\mathcal{J}$, family, such that:

- $\phi, X \in \mathcal{J}$
- $\forall_{U,V \in \mathcal{J}} U \cap V \in \mathcal{J}$
- $\forall_{\{U_s\}_{s \in S}}$ such that $\forall_{u_s \in \mathcal{J}}$, we have $\bigcup_{s \in S} U_s \in \mathcal{J}$

Note: $\mathcal{J}$ is a family of open sets U.

3.2
Base of Topology

A base of topology is a gamily $\mathcal{B}$ of sets $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ such that every set in $\mathcal{J}$ is a union of some sets from $\mathcal{B}$.

For example: Family of all open balls in metric space is a ball of that metric space.

Definition:

A topological space is called First Countable if it has a countable base, e.g., rational numbers, but not real numbers.

A topological space is called Second Countable if every point has a countable base of neighbourhoods.

3.3 COUNTABLITY:

$\mathbb{N}$ & $\mathbb{Q}$ is countable; i.e.,it can be passed like a sequence. But $\mathbb{R}$ is not countable.

DEFINTION:

A base of neighbourhoods of x is a family $\mathcal{B}_x$ of open sets such that:

- $x \in U$, $\forall U \in \mathcal{B}_x$
- $\forall_{x \in W \in \mathcal{J}} \exists_{U_W \in \mathcal{B}_x} U_W \subset W$, *small enough to be inside.*

DEFINITION:

A topological space is called Separable if it has a countable dense set.

THEOREM:

A metric space is firs countable $\iff$ it is separable.

3.4 BAIRE CATEGORY

LEMMA 1:

Let D - dense $\overline{D} = x$ in X and $U \subset X$ - open. Then $\overline{U} = \overline{U \cap D}$.

Proof.

We need to prove that $\overline{U} \subset \overline{U \cap D}$.

Let $x \in \overline{U}$. Then $\forall_{\epsilon>0} \exists_{y_\epsilon \in U} d(y_\epsilon, x) < \epsilon$ **(1)**

But D is dense in X. $\overline{D} = X$ so $\forall_{y \in X} \forall_{\epsilon>0} \exists_{z \in D} d(y, z) < \epsilon$ **(2)**

We need to prove that for $x \in \overline{U}$ we have $\forall_{\epsilon>0} \exists_{w \in (U \cap D)} d(x, w) < \epsilon$.

From **(1)** we have $y_\epsilon \in U$ such that $d(y_\epsilon, x) < \frac{\epsilon}{3}$. But since $y_\epsilon \in U$ and U is open, there is $r_\epsilon > 0$ such that $B(y_\epsilon, r_\epsilon) \subset U$. We can assume that $r_\epsilon < \frac{\epsilon}{3}$.

On the other hand, from **(2)** there is $z \in D$ such that $d(z, y_\epsilon) < \frac{\epsilon}{6}$. So let w = z and then:

$d(w, x) \leq d(x, y_\epsilon) + d(y_\epsilon, w) < \frac{\epsilon}{3} + \frac{\epsilon}{6} < \epsilon$.

LEMMA 2:

If A is nowhere dense then $x \setminus \overline{A}$ is dense in X.

Proof.

$X = \overline{X \setminus \overline{A}} \iff \phi = X \setminus \overline{(X \setminus \overline{A})} \iff \phi Int(X \setminus X \setminus \overline{A}) = Int(\overline{A})$. Recall: $X \setminus \overline{E} = Int(X \setminus E)$.

BAIRE CATEGORY THEOREM:

If (X,d) is a complete metric space and $\{A_n\}_{n\in\mathbb{N}}$ is family of nowhere dense, then $\bigcup_{n=1}^{\infty} A_n \neq X$.

To prove this theorem we will use the two lemmas that are mentioned above.

Proof.

Taking A_1. Let B_0 be a ball in X.

Since $X \setminus \overline{A_1}$ is dense in X, where $\overline{B_0} = \overline{B_0 \cap (X \setminus \overline{A})} = \overline{B_0 \setminus \overline{A}}$.

$B_0 \setminus \overline{A}$ is nonempty, ball is always not empty and so is the closure. because it is an open set.

There is an open ball B_1 such that $\overline{B_1} \subset B_0 \setminus \overline{A}$, is there smaller B?

$B_0 \setminus \overline{A_1}$ is open so there is a ball $B(x,r) \subset B_0 \setminus \overline{A_1}$ take $\overline{B}(x, \frac{r}{2}) \subset B(x,r)$,

and $\overline{B}(x,r) = \{y \in X : d(y,x) \leq R\}$, distance and balls are continuous.

$\overline{B}(x,R) = \{y \in X : d(x,y) \leq R\}$

$f : X \to \mathbb{R}$. f(z) = d(z,x), f is continuous.

We can assume that $diam(\overline{B_1}) < 1$ we can always shrink it.

$diam\overline{B}(x,r) < 2r$ take $x_1, x_2 \in \overline{B}(x,r)$. $d(x_1,x_2) \leq d(x_1,x) + d(x,x_2) \leq r + r = 2r$.

Take B_1 constructed as above, since $X \setminus A_2$ is dense in X.

$\overline{B_1} = \overline{B_1 \setminus \overline{A}}$, ... we get balls B_i such that $diam\overline{B_i} \subset B_{i-1} \setminus \overline{A_i}$.

Family $\{\overline{B_i}\}$ of closed sets of descending diameters is non-empty intersection, because X is complete

$x_0 \in \bigcap_{i=1}^{\infty} \overline{B_i}$.

$x_0 \in \bigcap_{i=1}^{\infty} \left(B_{i-1} \setminus \overline{A_i}\right) \subset B_0 \cap \bigcap_{i=1}^{\infty} \left(X \setminus \overline{A_i}\right)$ intersection of complements

$= B_0 \cap \left(X \setminus \bigcup_{i=1}^{\infty} \overline{A_i}\right.$ non-empty.

So for every ball B_0 in X $B_0 \cap \left(X \setminus \bigcup_{i=1}^{\infty} \overline{A_i}\right) \neq \phi$.

So, $X \setminus \bigcup_{i=1}^{\infty} \overline{A_i}$ is dense in X.

3.5 Connectedness

Lemma:

A space is connected if it is not a union of the disjoints open sets.

Equivalently: x cannot be presented as $x \in F_1 \cup F_2$ where F_1 , F_2 are closed and disjoints.

Equivalently: x cannot be presented as $x = A_1 \cup A_2$ where A_1 , A_2 are seperated sets, i.e., $A_1 \cap \overline{A_2} = \phi = \overline{A_1} \cap A_2$.

Equivalently: Every continuous function from x tp {0,1} with discrete metric is constant.

Lemma:

A path in X from x to y is continuous function. from $\mathbb{R}$ with its usual metric.

Lemma:

$\gamma : [0, 1] \to X$ suhc that $\gamma(0) = x$ and $\gamma(1) = u$.

So a space is pathe connected if for every two points x, y there is a path from x to y.

Theorem:

If x is path connected then it is connected.

If $\{C_s\}_{s \in \mathcal{S}}$ is a family of connected sets, and $\left(C_{\mathcal{S}_0}, C_{\mathcal{S}}\right)$ are not separated for all $s \in \mathcal{S}$, then $\bigcup_{s \in \mathcal{S}} C_{\mathcal{S}}$ is connected. The proof is very technical and not needed for further topics.

Theorem:

If $f : x \to y$ is continuous and x is connected then f(x) is connected.

Note: There is a connected space that is not path connected.

E.g., $X = \{\text{Graph of } \sin \frac{1}{x}\} \cup \{\{0\} \times [-1.1]\} \, in \, \mathbb{R}^2$ and notice that the graph changes fast, e.g., 2000π , $2000 + 2\pi$, and the graph is decomposed but not separate.

DEFINTION:

Let $f, g : [0,1] \to X$

f(0) = x =g(0)

f(1) = y =g(1)

Homotopy between f and g is a function map $H : [0,1] \ni t \times [0,1] \ni s \to X$.

$H(t,s) \in X$ such that $\forall_t H(t,0) = H_0(t) = f(t)$, and $H(t,1) = H_1(t) = g(t)$.

NOTES:

- Every metric space is second countable, then $\mathcal{B}_x = \{B(x, \frac{1}{n}), n \in \mathbb{N}\}$
- A neighbourhood of x is any open set contains x.
- $$Topological\ Base = \begin{cases} x \in U \cap V \\ \exists x \in W \subset U \cap V \end{cases}$$
- $y \in Int(\overline{C}) \iff \exists_{r>0} B(y,r) \subset \overline{C}$
- $Int(\overline{C}) = \phi \implies \forall_{z \in \overline{C}}\, z \notin Int(\overline{C}) \iff {}_{z \in \overline{C}} \forall_{r>0}\, B(z,r) \not\subset \overline{C}$ $\iff {}_{z \in \overline{C}} \forall_{r>0} \exists_{w \in \overline{C}}\, w \in \overline{C} \setminus B(z,r) = \exists_{w \in X} w \in \left(X \setminus \overline{C}\right) \bigcap B(z,r)$ which both are open.
- Topology is about neighbourhoods.
- Cantor set is nowhere dense, but the question is how many neighbors do we need.
- $U \cap D \subset U \iff \overline{U \cap D} \subset \overline{U}$
- A set $A \subset X$ is nowhere dense if $Int(\overline{A}) = \phi$
- Ball is always not empty and so is the closure.
- Distance and balls are continuous.
- Category is first countable and everything else is second countable.
- A loop is a path from x to x.
- $\text{Loop}_1 \sim \text{Loop}_2 \iff$ there is a homotopy between them.
- A loop is contractable of if it is homotopic to a constant loop, e.g., f(t) = x.
- Loops has properties like notation for directions by +, -, and decomposition.
- **Currently most of mathematicians work is concentrated about Algebraic Topology.**

IV

Exercises

Everywhere below (X,d) is a metric space. The following definitions were given in the previous chapters.

Let $A \subset X$. We define the *diameter* of the set A by the formula:

$$diam(A) = \sup_{x,y \in A} d(x, y).$$

If diam(A) is finite, we say that the set A is bounded.

For all $x \in X$ we define the *distance of x from the set A* by the formula:

$$dist(x, A) = inf_{y \in A} d(x, y).$$

Let (X,d) be a metric space and let (x_n) be a sequence of elements (points) of X. We say that (x_n) is *convergent* to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$. We write then $x_n \longrightarrow x$.

Let (X,d) be a metric space and let $F \subset X$. We say that the set F is *closed* if $X \setminus F$ is an open set. The following statement was proved in the previous chapters: F is closed if and only if for all sequences x_n of points of the set F such that $x_n \longrightarrow x$ for some $x \in X$, we have $x \in F$. In other words: the set F is closed if and only if every convergent sequence of points of F is convergent to a point of F.

Let $A \subset X$. We say that $x_0 \in X$ is a *cluster point of A* if there exists a sequence x_n such that $(x_n) \subset A$, $\forall_{n \in \aleph} x_n \neq x_0$ and $x_n \longrightarrow x$. We denote by A^d the set of all cluster points of A.

For a set $A \subset X$ we define its *closure* $\overline{A}$ by the formula $\overline{A} = A \cup A^d$

Let (X,d) be a metric space. A set $U \subset X$ is called *open*, if:

$$\forall_{x \in U} \exists_{r_x > 0} B(x, r_x) \subset U$$

Let B(x,r) be an open ball and let $\overline{B}(x, r)$ be a closed ball in a metric space (X,d). Recall that $B(x, r) = \{y \in X : d(x, y) < R\}$ and $\overline{B}(x, r) = \{y \in X : d(x, y) \leq r\}$.

Let (X,d) and (Y, ρ) be metric spaces. We say that a function $f : X \to Y$ is continuous at a point $x_0 \in X$if

$$\forall_{\epsilon>0} \exists_{\delta>0} \forall_{x\in X} \, d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \epsilon.$$

We say that f is continuous, if f is continuous at every point of its domain.

Recall that the "river distance function" d_R on $\mathbb{R}^2$ is given by:

$$d_R(x, y) = \begin{cases} 0 & if\, x = y \\ |x_2| + |y_2| + |x_1 - y_1| & if\, x \neq y \end{cases}$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

Recall that the "railway distance function" d_r on $\mathbb{R}^2$ is given by:

$$d_r(x, y) = \begin{cases} 0 & if\, x = y \\ \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2} & if\, x \neq y \end{cases}$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

Exercise 1 Prove that d_R and d_r are distance functions on $\mathbb{R}^2$.

Exercise 2 Draw on a plane $\mathbb{R}^2$ the following open balls: B((0,0),1),B((1,2),3),B((1,2),3),B((1,2),6), and $B((1, 1), \frac{1}{2})$, with respect to distance functions d_R and d_r.

Exercise 3 Let $X = [0, 1] \cup \{2\}$ and let

$$d_a(x, y) = \begin{cases} |x - y| & if\, x, y \in [0, 1] \\ a & if\, x = 2 \text{ and } y \neq 2 \\ a & if\, y = 2 \text{ and } x \neq 2 \\ 0 & if\, x = 2 \text{ and } y = 2 \end{cases}$$

Check if d_a is a distance function on X for $a = 1$, $a = \frac{1}{2}$, $a = \frac{1}{3}$.

Exercise 4 Let (X,d) be a metric space and let $Y \subset X$. Let $d_{|Y\times Y}$ be the restriction of d to Y $\times Y, i.e.,$d$_{|Y\times Y}(x, y) = d(x, y)$ for all $x, y \in Y$.

Prove that $d_{|Y\times Y}$ is a distance function on Y, i.e., $(Y, d_{|Y\times Y})$ is a metric space.

Exercise 5 Let (X,d) be a metric space, let $x \in X$ and let r > 0. Prove that an open ball B(x,r) is an open set.

Exercise 6 Let U and V be open sets. Prove that $U \cup V$ is open.

Exercise 7 let $X = \mathbb{R}$ and let d be the natural distance on $\mathbb{R}$, given by $d(x, y) = |x - y|$. Check if the following sets are open in (X,d):

1. $(0, 1)$
2. $[0, 1)$
3. $[0, 1]$

Exercise 8 Let $X = [0, +\infty)$, and let d be the distance function induced on X from the natural distance function on $\mathbb{R}$, given by $d(x, y) = |x - y|$. Check if the following sets are open in (X,d):

1. $[0, 1)$
2. $[0, 1]$

Exercise 9 Let (X,d) be a discrete space, i.e., $X \neq \phi$ and

$$d(x, y) = \begin{cases} 1 & if x \neq y \\ 0 & if x = y \end{cases}$$

for all $x, y \in X$. Check if the set $\{z\}$ is open. Are there any subsets of X that are not open?

Exercise 10 Prove that A is bounded set if and only if there exists a ball B(x,r) such that $A \subset B(x, r)$. Hint: try to estimate r using diam(A).

Exercise 11 Prove that diamB(x,r) $\leq diam\overline{B}(x, r)$ and $diam\overline{B}(x, r) < 2r$. Hint: consider the discrete space where d(x,y) = 1 for $x \neq y$ and d(x,y) = 0 for x = y.

Exercise 12 Give an example of a set $A \subset X$ and a point $x \in X$ such that dist(x,A) < d(x,y) for all $y \in A$.

Exercise 13 Prove that if $x \in A$ then dist(x,A) = 0.

Exercise 14 Give an example of a set $A \subset X$ and a point $x \in X$ such that dist(x,A) = 0 and $x \notin A$.

Exercise 15 Give an example of a set $A \subset X$ and a point $x \in X$ such that dist(x,A) = d(x,y) for:

- all y inA.
- a single point $y \in A$.
- exactly 3 points $y \in A$.

Exercise 16 Let $x \in X$. Prove that $\{x\}$ is a closed set.

Exercise 17 Prove that if F_1, F_2 are closed sets then $F_1 \cup F_2$ and $F_1 \cap F_2$ are closed.

Exercise 18 Prove that if $\{F_s\}$ is a family of closed sets, then the intersection $\bigcap_s F_s$ is a closed set.

Exercise 19 Prove that if F is a compact set, then for every $x \in X$ there exists $y \in F$ such that dist(x,F) = d(x,y).

Exercise 20 Let $A \subset X$ and let $x \in X$. Prove that $dist(x, A) = dist(x, \overline{A})$.

Exercise 21 Prove that for all $A \subset X$ we have $diam(A) = diam(\overline{A})$.

Exercise 22 Prove that $\overline{B(x, r)} \subset \overline{B}(x, r)$. Recall that in the discrete space we have $\overline{B(x, r)} \neq \overline{B}(x, r)$.

Exercise 23 Prove that a function $f : X \to Y$ is continuous at a point $x_0 \in X$ if and only if for every sequence (x_n) such that $x_n \to x_0$ we have $f(x_n) \to f(x_0)$.

Exercise 24 Let (X,d) be a metric space and let $y \in X$. Prove that the function $f_y(x) = d(x, y)$ is continuous. Consider f_y as a function $f_y : X \to \mathbb{R}$, where $\mathbb{R}$ has its natural metric ρ given by $\rho(a, b) = |a - b|$ for all $a, b \in \mathbb{R}$.

Exercise 25 Let (X,d) be a metric space and let $A \subset X$. Prove that the function $f_A(x) = dist(x, A)$ is continuous.

Exercise 25 Let $X = \{1, 2, 3\}$ and let $d : X \times X \to [0, +\infty)$ be function defined by formulas:
$d(1, 1) = d(2, 2) = d(3, 3) = 0,$
$d(1, 2) = d(2, 1) = d(1, 3) = d(3, 1) = 1.$
$d(2, 3) = d(3, 2) = 3.$
Check if d is a distance function on X.

Exercise 26 Let $||f||_0 = |f(0)| + \max_{x\in[0,1]}|f(x)|$. Prove that $||.||_0$ is a norm on the space $C[0, 1]$, i.e., that for every real-valued coninuous function $f : [0, 1] \to \mathbb{R}$ we have:

1. $||f||_0 \geq 0.$
2. $||f||_0 = 0 \iff \forall_{x\in[0,1]} f(x) = 0$
3. $||a.f||_0 = |a|.||f||_0$ for all $a \in \mathbb{R}$
4. $||f + g||_0 \leq ||f||_0 + ||g||_0$ for all $f, g \in C \in [0, 1]$

Exercise 27 Compute diameter diam(A) of the 4-element set $A = \{(0, 0), (1, 1), (1, 2), (2, 2)\}$ in the metric space $(\mathbb{R}^2, d)$, where d is defined by the formula:

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2| & if\, x_1 = x_2 \\ |x_1 - x_2| + |y_1| + |y_2| & if\, x_1 \neq x_2 \end{cases}$$

Exercise 28 Let (X,d) be a metric space and let $x, y \in X$. Show that the set

$$A = \{z \in X : d(x, z) < d(y, z)\}$$

is open in (X,d).

Hint: Let $z_0 \in A$, then $d(x, z_0) < d(y, z_0)$. How small must be $d(z, z_0)$ so that $d(x, z) < d(y, z)$ (use triangle inequalities with z, z_0, x, y to estimate each side if this inequality)?

Exercise 29 Check if the set $\{p_n\}_{n\in\mathbb{N}}$ of points $p_n = (\frac{1}{n}, \frac{1}{n})$ is closed in the metric space $(\mathbb{R}^2, d_R)$. where d_R is the river metric.

Hint: check if (0,0) is the limit of the sequence $\{p_n\}$ in the metric space $(\mathbb{R}^2, d_R)$.

Exercise 30 Let d(x,y) = |x-y| and let

$$d_0(x, y) = \begin{cases} 1 & if x \neq y \\ 0 & if x = y \end{cases}$$

Check if the function $f : (\mathbb{R}, d) \to (\mathbb{R}, d_0)$ given by the formula

$$f(x) = \begin{cases} 1 & if x > 0 \\ 0 & if x = 0 \\ -1 & if x < 0 \end{cases}$$

is continuous at points x = 2 and x = 0.

Exercise 31 Recall that the space $B(\mathbb{R})$ of all bounded, continuous real-valued functions on $\mathbb{R}$ is a metric space with metric $\rho(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$ for $f, g \in B(\mathbb{R})$.

Find a sequence of continuous functions f_n such that for all $n \in \mathbb{N}$ we have $\sup_{x \in \mathbb{R}} |f_n(x)| < 1$, and the sequence $\{f_n\}$ has no convergent subsequence in $(B(\mathbb{R}, \rho)$.

Exercise 32 Let L be the set of all linear functions on [0,1], i.e.,

$$f \in L \iff \exists_{a,b \in \mathbb{R}} \forall_{x \in [0,1]} f(x) = a.x + b.$$

Let $\rho(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. Recall that $(C[0, 1], \rho)$ is the metric space of all real-valued continuous functions on interval [0,1], with metric space ρ.

1. Prove that L is not open in $(C[0, 1], \rho)$, i.e., for every $\epsilon > 0$ and every linear function f(x) = a.x+b there exists a continuous function g which is not linear and we have $\rho(f, g) < \epsilon$.
2. Prove that $f_n \in L$ for all $n \in \mathbb{N}$ and there exists a continuous function f such that for all $x \in [0, 1]$ we have $f_n(x) \to f(x)$ then f is linear and $\rho(f_n, f) \to 0$.

   Hint: coefficients a_n and b_n of every linear f_n are determined by two values of f_n at some points of the interval [0,1].
3. Prove that L is closed n $(\mathbb{C}[0, 1], \rho)$,i.e., if $f_n \in L$ for all $n \in \mathbb{N}$ and there exists a continuous function f sucht tha $\rho(f_n, f) \to 0$, then f is linear.

Exercise 33 Let d_* be a metric on $\mathbb{R}^2$ defined as follows:

$$d_*((x_1, y_1), (x_2, y_2)) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}.$$

Show that $(\mathbb{R}^2, d_*)$ is a complete metric space (you don't need to prove that it is a metric space, prove only that it is complete).

Hint: you may need to use completeness of real numbers with their standard metric.

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